HOMEWORK 10, CALCULUS AND LINEAR ALGEBRA, 2015/2016

Assigned 12/02/2015, due 12/09/2015, collected from 2pm to 2.15pm sharp!

Name and Family Name (CAPITAL LETTERS): ______________________

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Exercise 1

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}$.

a) Compute the determinant of A using the Laplace method.

b) If possible, compute the inverse matrix of A.

c) Using the matrix method, solve the following linear system

$$\begin{cases} x + z = 1 \\ -2x + y = -1 \\ 3x + 2y = 0 \end{cases}$$

Solution:

a) It is convenient to choose the third column to apply the Laplace algorithm for the determinant (becausethere are a lot of zeros...) :

$$\det A = 1 \cdot \det \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} = -4 - 3 = -7 .$$

b) Denote by A_{ij} the determinant of the 2x2 matrix obtained from A by removing the row i and the column j. Then:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} 0 & 2 & -1 \\ 0 & -3 & -2 \\ -7 & -2 & 1 \end{pmatrix}$$

c) Observe that the coefficient matrix of the linear system is A. Set $b = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$AX = b \iff X = A^{-1}b = -\frac{1}{7} \begin{pmatrix} 0 & 2 & -1 \\ 0 & -3 & -2 \\ -7 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} .$$

Namely $x = \frac{2}{7}$, $y = \frac{-3}{7}$, $z = \frac{5}{7}$.

Exercise 2

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

a) Compute the eigenvalues of A.

b) Compute the associated eigenvectors with length 1.

c) Consider the vector $\vec{w} = \frac{2}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{17}}\begin{pmatrix}2\\3\\-2\end{pmatrix}$. Compute the product $A\vec{w}$ using the previous results.

Solution:

a) λ is an eigenvalue of A if and only if

$$\det(A-\lambda I) = 0 \iff \det\begin{pmatrix}1-\lambda & 2 & 1\\ 6 & -1-\lambda & 0\\ -1 & -2 & -1-\lambda\end{pmatrix} = 0 \iff$$
$$\iff -12-(1+\lambda)-(1+\lambda)\big(-(1-\lambda)(1+\lambda)-12\big) = 0 \iff$$
$$\iff -\lambda(\lambda+4)(\lambda-3) = 0\ .$$

Therefore the eigenvalues of A are $\lambda = 0, -4, 3$.

b) $\vec{v} \neq 0$ is an eigenvector of A associated to the eigenvalue λ if and only if $A\vec{v} = \lambda\vec{v}$, that is $(A-\lambda I)\vec{v} = 0$.

For the eigenvalue $\lambda = 0$, the eigenvectors $\vec{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ are the non-zero solutions of

$$A\vec{v} = 0 \iff \begin{cases}x+2y+z=0\\ 6x-y=0\\ -x-2y-z=0\end{cases} \iff \begin{cases}y=6x\\ z=-13x\end{cases} \iff \vec{v} = \begin{pmatrix}x\\6x\\-13x\end{pmatrix}\ .$$

The length of $\vec{v}$ is one if you choose $x = 1/\sqrt{206}$.

For the eigenvalue $\lambda = -4$, the eigenvectors $\vec{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ are the non-zero solutions of

$$(A+4I)\vec{v} = 0 \iff \begin{cases}5x+2y+z=0\\ 6x+3y=0\\ -x-2y+3z=0\end{cases} \iff \begin{cases}y=-2x\\ z=-x\end{cases} \iff \vec{v} = \begin{pmatrix}x\\-2x\\-x\end{pmatrix}\ .$$

The length of $\vec{v}$ is one if you choose $x = 1/\sqrt{6}$.

For the eigenvalue $\lambda = 3$, the eigenvectors $\vec{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ are the non-zero solutions of

$$(A-3I)\vec{v} = 0 \iff \begin{cases}-2x+2y+z=0\\ 6x-4y=0\\ -x-2y-4z=0\end{cases} \iff \begin{cases}y=\frac{3}{2}x\\ z=-x\end{cases} \iff \vec{v} = \begin{pmatrix}x\\\frac{3}{2}x\\-x\end{pmatrix}\ .$$

The length of $\vec{v}$ is one if you choose $x = 1/\sqrt{6}$.

c) By point b), $\begin{pmatrix}-1\\2\\1\end{pmatrix}$ is an eigenvector of A associated to the eigenvalue -4 (set $x=-1$) and $\begin{pmatrix}2\\3\\-2\end{pmatrix}$ is an eigenvector of A associated to the eigenvalue 3 (set $x=2$). Therefore

$$A\vec{w} = \frac{2}{\sqrt{6}}A\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{17}}A\begin{pmatrix}2\\3\\-2\end{pmatrix} = \frac{2}{\sqrt{6}}(-4)\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{17}}3\begin{pmatrix}2\\3\\-2\end{pmatrix} = \begin{pmatrix}\frac{8}{\sqrt{6}}+\frac{6}{\sqrt{17}}\\ -\frac{16}{\sqrt{6}}+\frac{9}{\sqrt{17}}\\ -\frac{8}{\sqrt{6}}-\frac{6}{\sqrt{17}}\end{pmatrix}\ .$$